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A Convergence Result for a Parallel Algorithm for Solving the Navier-Stokes Equations

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Abstract—This work is concerned with the convergence/stability analysis of a parallel algorithm which is used to solve the incompressible Navier-Stokes problem. This relies on a splitting of the main differential operator, thanks to which the most important difficulties (nonlinearity and incompressibility) can be considered independently. Thus, one is led to the formulation of subproblems of two kinds which can be solved simultaneously by two different processors. We prove conditional stability and convergence; this can serve to justify the accuracy of previous numerical results.

Keywords—Numerical simulation, Computational fluid dynamics, Parallel algorithm, Navier-Stokes equations, Splitting methods.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a regular and bounded open region ($n = 2$ or 3). In fluid mechanics, the description of a homogeneous incompressible fluid whose particles are located in Ω during the time interval $[0, T]$ is given by the knowledge of its velocity field $u = u(x, t)$ and two thermodynamical variables, for instance, the pressure and temperature distributions, $p = p(x,t)$ and $v = v(x, t)$, respectively. In the case of a viscous fluid, u and p solve the so-called Navier-Stokes equations:

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \qquad & \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0, \qquad & \text{in } \Omega \times (0,T). \end{aligned} \tag{1}$$

Here, $\nu > 0$ is the kinematic viscosity of the fluid and $f = f(x, t)$ is the density function for a field of external forces. For simplicity, we have assumed in (1) that the fluid possesses unit constant density. Once u and p are found, v can be computed by solving an appropriate transport-diffusion problem.

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The solution of (1) involves several major difficulties, namely, the following.

- We are concerned with a system of PDEs and not with a single PDE.
- The unknowns u_i are coupled through the incompressibility condition $\nabla \cdot u = 0$.
- The problem is nonlinear, owing to the presence of the term $(u \cdot \nabla)u$.

As a consequence, we must not expect to obtain exact solutions, but only for a few special situations. This has fostered the research of numerical algorithms providing approximate solutions.

Usually, the numerical approximation procedure consists of two major steps. First, one discretizes with respect to the time variable t and then the resulting (sub)problems are discretized in space. There are many classical sequential schemes that can be used to discretize in time [1]. Among the most frequently used, we find the so-called *fractional step methods* (see [2]). This paper is devoted to the convergence analysis of a parallel algorithm which is also obtained by discretizing in time with a fractional step technique. This was introduced in [3] by the authors.

In order to better understand the underlying philosophy of fractional step methods and their connections with parallelization, consider the following model problem:

$$\frac{du}{dt} + Au = f, \qquad t > 0, \quad u(0) = u_0.$$

Here, $u = u(t)$ (scalar or vector) is the unknown and $f = f(t)$ is given. We assume that A (linear or nonlinear) is an operator defined in an appropriate vector space. If the time interval is assumed to be divided into subintervals of amplitude Δt, then a "natural" implicit method for the numerical approximation to a solution u is given as follows:

$$\frac{u^{m+1} - u^m}{\Delta t} + Au^{m+1} = f\left((m+1)\Delta t\right) = f^{m+1}, \qquad m \geq 0. \tag{2}$$

Once (2) is solved, u^{m+1} is, at least formally, an approximation to the solution u at time $(m+1)\Delta t$.

Let us now assume that A can be split in the form

$$A = A_1 + A_2,$$

where each A_i ($i = 1, 2$) is a new operator. Suppose that, for some reason, it is easier to solve (2) if A is replaced by A_i. Then, a fractional step method (of the Peaceman-Rachford type) that can be applied reads as follows. For given $m \geq 0$ and u^m, first compute $u^{m+1/2}$ by solving

$$\frac{u^{m+1/2} - u^m}{\Delta t/2} + A_1 u^{m+1/2} + A_2 u^m = f^{m+1/2}. \tag{3}$$

Then, in a second step, compute u^{m+1} by solving

$$\frac{u^{m+1} - u^{m+1/2}}{\Delta t/2} + A_1 u^{m+1/2} + A_2 u^{m+1} = f^{m+1}. \tag{4}$$

In (3) and (4), we may take (for instance) $f^{m+i/2} = f((m + i/2)\Delta t)$ for $i = 1, 2$.

The previous algorithm is purely sequential. Starting from a given $u^0 = u_0$, $u^{1/2}$ is calculated from (3). Next, u^1 is obtained from (4) and so on. It is, however, easy to parallelize algorithm (3),(4). To this purpose, we compute u^{m+1} in three rather than two steps: $u^{m+2/3}$ and $u^{m+4/3}$ are calculated simultaneously (with two different processors) by solving

$$\frac{u^{m+2/3} - u^m}{2\Delta t/3} + A_1 u^{m+2/3} + A_2 u^m = f^{m+2/3},$$
$$\frac{u^{m+4/3} - u^m}{4\Delta t/3} + A_1 u^m + A_2 u^{m+4/3} = f^{m+4/3},$$

and then u^{m+1} is obtained from the formula

$$u^{m+1} = \frac{1}{2}\left(u^{m+2/3} + u^{m+4/3}\right). \tag{5}$$

Obviously, this can be generalized to the case where A can be written in the form

$$A = A_1 + A_2 + \cdots + A_q.$$

This requires the use of q processors in parallel in a scheme involving $q+1$ fractional steps.

In [3], we have adapted the above ideas to the numerical solution of (1) together with appropriate boundary and initial conditions for u by introducing a three-steps scheme which, therefore, involves the simultaneous solution of two subproblems. As in [2], the splitting of the "spatial" differential operator (the equivalent of A in (1)) separates the main difficulties, namely, nonlinearity and incompressibility. More precisely, for each m, one has to compute simultaneously the solution $u^{m+2/3}$ to a quasi-linear elliptic system and the solution $\{u^{m+4/3}, p^{m+4/3}\}$ to a linear quasi-Stokes problem. Then, u^{m+1} is calculated from (5).

Before recalling the formulation of our algorithm properly, it is convenient to introduce the following.

(a) $\mathcal{D}(\Omega) = \{\varphi \in C^\infty(\Omega);\ \operatorname{Supp}\varphi \subset \Omega\}$.

(b) $H^1(\Omega) = \{v \in L^2(\Omega);\ \nabla v \in L^2(\Omega)^n\}$, a Hilbert space for the norm

$$\|v\|_{H^1} = \left(\int_\Omega |v|^2\,dx + \int_\Omega |\nabla v|^2\,dx\right)^{1/2}.$$

(c) $H_0^1(\Omega) =$ the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$; in $H_0^1(\Omega)$, the seminorm

$$\|v\|_{H_0^1} = \left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2}$$

is in fact a norm, equivalent to the norm of $H^1(\Omega)$.

(d) $J(\Omega) = \{\varphi \in \mathcal{D}(\Omega)^n;\ \nabla\cdot\varphi = 0 \text{ in } \Omega\}$.

(e) $V =$ the closure of $J(\Omega)$ in $H_0^1(\Omega)^n$; V is a Hilbert space for the scalar product and norm of $H_0^1(\Omega)^n$, which will be denoted by $((.,.))$ and $\|.\|$, respectively.

(f) $H =$ the closure of $J(\Omega)$ in $L^2(\Omega)^n$; H is a new Hilbert space for the scalar product and norm of $L^2(\Omega)^n$, denoted by $(.,.)$ and $|\,.\,|$.

(g) $V' =$ the dual of V; $\langle .,. \rangle$ denotes the duality pairing between V' and V.

(h) We also introduce the trilinear forms $b(\cdot,\cdot,\cdot)$ and $\hat{b}(\cdot,\cdot,\cdot)$:

$$b(u,v,w) = \int_\Omega u_i D_i v_j w_j\,dx, \qquad \hat{b}(u,v,w) = \frac{1}{2}\left(b(u,v,w) - b(u,w,v)\right),$$

for $u, v, w \in H^1(\Omega)^n$ (here, the usual summation convention is used).

The following properties of V and H are well known:

$$V = \left\{v \in H_0^1(\Omega)^n;\ \nabla\cdot v = 0 \text{ in } \Omega\right\},$$
$$H = \left\{v \in L^2(\Omega)^n;\ \nabla\cdot v = 0 \text{ in } \Omega,\ v\cdot\vec{n} = 0 \text{ on } \Gamma\right\},$$
$$V \hookrightarrow H \hookrightarrow V', \text{ where the embeddings are dense and compact.}$$

We can now give a rigorous formulation of the nonstationary Navier-Stokes problem in $\Omega\times(0,T)$.

To find a function $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that

$$\begin{gathered}\left\langle \frac{du(t)}{dt}, v\right\rangle + \nu\left((u(t),v)\right) + b\left(u(t),u(t),v\right) = \langle f(t), v\rangle,\\ \forall v \in V, \qquad t \text{ a.e. in } (0,T),\\ u(0) = u_0.\end{gathered} \tag{6}$$

Here, $u_0 \in H$ is a prescribed initial velocity field, f is a given function in $L^2(0,T;L^2(\Omega)^n)$, and $\nu > 0$ is a constant. It is well known that problem (6) possesses at least one solution, which is furthermore unique if $n = 2$. If u is a solution, then u solves, together with some scalar function p, the Navier-Stokes equations (1). One also has $u = 0$ on $\Gamma \times (0,T)$ and $u|_{t=0} = u_0$, in an appropriate sense.

An important property of $b(\cdot,\cdot,\cdot)$ is that $b(u,v,v) = 0$ for all $v \in V$, which is not true in general if $\nabla \cdot v \neq 0$. This suggests to replace (as in [1]) the nonlinear term $(u\cdot\nabla)u$ by

$$(u\cdot\nabla)u + \frac{1}{2}(\nabla\cdot u)u,$$

which is associated by the trilinear form $\hat{b}(u,u,v)$. Of course,

$$b(u,u,v) = \hat{b}(u,u,v), \qquad \forall\, u \in V, \quad \forall\, v \in H_0^1(\Omega)^n,$$

and also, $\hat{b}(u,v,v) = 0$, $\forall\, v \in H_0^1(\Omega)^n$ (even when $\nabla\cdot v \neq 0$). Obviously, the variational equation in (6) can also be written in terms of $\hat{b}(\cdot,\cdot,\cdot)$:

$$\begin{gathered}\left\langle \frac{du(t)}{dt}, v \right\rangle + \nu\left((u(t),v)\right) + \hat{b}\left(u(t),u(t),v\right) = \langle f(t), v\rangle, \\ \forall\, v \in V, \qquad t \text{ a.e. in } (0,T).\end{gathered} \tag{7}$$

2. THE ALGORITHM

As mentioned above, the numerical solution of (6) involves two main difficulties: the presence of the nonlinear term $(u\cdot\nabla)u$ and the incompressibility condition $\nabla\cdot u = 0$. A high performance strategy consists of introducing fractional step (alternate direction) discretization in time. This enables us to surmount these difficulties separately (see [6] and the references therein). A related parallel algorithm, introduced in [3] by the authors, also leads to good numerical results. As we said in Section 1, the corresponding convergence/stability analysis is the main subject of this paper. The numerical approximation is carried out at two levels.

(a) *Approximation with respect to the time variable t*—first, the time derivatives are replaced by difference quotients. Accordingly, one is led to the formulation of stationary subproblems of two kinds where only one of the above mentioned difficulties is conserved. At each time step, two of these subproblems can be solved simultaneously. Their solutions can then be used to compute an approximation at the next value of t.

(b) *Approximation with respect to space variables x_i*—this is needed to solved the previous subproblems. Among other possibilities, we have used in [3] finite element methods, although the theoretical arguments in Sections 3 and 4 also hold for other schemes.

Let us describe the algorithm more in detail.

FIRST LEVEL. Assume $[0,T]$ is divided in N subintervals of length k ($k = T/N$) and let $\theta \in (0,1)$ be given. First, set

$$u^0 = u_0. \tag{8}$$

Then, for given $m \geq 0$ and $u^m \in H_0^1(\Omega)^n$, compute u^{m+1} as follows.

PROBLEM (NLP). Solve the nonlinear elliptic system

$$\begin{gathered}\frac{1}{2k/3}\left(u^{m+2/3} - u^m, v\right) + \nu\left(\left(\theta u^m + (1-\theta)u^{m+2/3}, v\right)\right) \\ +\hat{b}\left(u^{m+2/3}, \theta u^m + (1-\theta)u^{m+2/3}, v\right) = \left(f^{m+2/3}, v\right), \\ \forall\, v \in H_0^1(\Omega)^n, \qquad u^{m+2/3} \in H_0^1(\Omega)^n.\end{gathered} \tag{9}$$

PROBLEM (LP). Solve the linear (quasi-Stokes) problem

$$\begin{aligned}&\frac{1}{4k/3}\left(u^{m+4/3}-u^m,v\right)+\nu\left(\left(\theta u^{m+4/3}+(1-\theta)u^m,v\right)\right)\\&=\left(f^{m+4/3},v\right)-\hat{b}\left(u^m,u^m,v\right),\qquad\forall v\in V,\quad u^{m+4/3}\in V,\end{aligned}\tag{10}$$

and, finally, set

$$u^{m+1}=\frac{1}{2}\left(u^{m+2/3}+u^{m+4/3}\right).\tag{11}$$

In (9) and (10), $f^{m+i/3}$ is given by

$$f^{m+i/3}=\frac{3}{ik}\int_{mk}^{(m+i/3)k}f(x,t)\,dt,\qquad\text{for } i=2,4.$$

Obviously, each $u^{m+i/3}$ is, at least formally, an approximation to the function $u(\cdot,(m+i/3)k)$. As announced, we see that, after discretization in time, the task is reduced to the solution of stationary problems of two kinds: quasilinear elliptic problems (NLP)—where the incompressibility condition has disappeared—and linear problems (LP). Existence and uniqueness results for (LP) and (NLP) can be deduced in a relatively simple manner, at least when ν is not too small, with standard arguments (see, e.g., [4,5]).

SECOND LEVEL. Let us assume that a Hilbert external approximation is given for $H_0^1(\Omega)^n$ (see [1]). This means that we have at our disposal a family of triplets

$$\{(W_h,p_h,r_h)\}_{h\in\mathcal{H}}$$

(here, $\mathcal{H}$ is a generalized sequence in $\mathbb{R}^q$ that converges to zero), a Hilbert space F, and an isomorphism $\bar{\omega}$ from $H_0^1(\Omega)^n$ onto F such that, for each $h\in\mathcal{H}$, one has the following:

(i) W_h is a finite-dimensional space with scalar product $((\cdot,\cdot))_h$ and norm $\|\cdot\|_h$,
(ii) $p_h:W_h\to F$ is a bounded linear operator,
(iii) $r_h:H_0^1(\Omega)^n\to W_h$ is a (possibly discontinuous) mapping.

We assume that the previous external approximation is

(A1) *conformal* in $L^2(\Omega)^n$, i.e., $W_h\subset L^2(\Omega)^n$, $\forall\, h\in\mathcal{H}$,
(A2) *stable*, i.e., $\|p_h\|_{\mathcal{L}(W_h;F)}$ is uniformly bounded,
(A3) *convergent*, i.e.,
  (A3.a) $p_h r_h w\to\bar{\omega}w$ strongly in F as $h\in\mathcal{H}$, $h\to 0$ for all $w\in H_0^1(\Omega)^n$, and also
  (A3.b) if $\mathcal{H}'\subset\mathcal{H}$, $\mathcal{H}'\to 0$, $w_{h'}\in W_{h'}$ for all $h'\in\mathcal{H}'$ and $p_{h'}w_{h'}\to\psi$ weakly in F, then $\psi\in\bar{\omega}(H_0^1(\Omega)^n)$.

On the other hand, let V_h be, for each $h\in\mathcal{H}$, a subspace of W_h and assume there exist mappings $s_h:V\to V_h$ such that

(A4) $s_h w\to w$ strongly in $L^2(\Omega)^n$ as $h\in\mathcal{H}$, $h\to 0$, $\forall\, w\in J(\Omega)$,
(A5) $\{(V_h,p_h|_{V_h},s_h)_{h\in\mathcal{H}},\bar{\omega}|_V,F\}$, regarded as an external approximation of V, is stable and convergent.

For each $h\in\mathcal{H}$, let $\hat{b}_h:W_h\times W_h\times W_h\to\mathbb{R}$ be a given trilinear form. We discretize (8)–(11) as follows. First, u_h^0 is the orthogonal projection of u_0 in V_h for the scalar product in $L^2(\Omega)^n$, i.e.,

$$\left(u_h^0,v_h\right)=(u_0,v_h),\qquad\forall\, v_h\in V_h,\quad u_h^0\in V_h.\tag{12}$$

For given $m\geq 0$ and $u_h^m\in W_h$, then the following.

1. We compute a solution $u_h^{m+2/3}\in W_h$ of the nonlinear problem

$$\begin{aligned}&\frac{1}{2k/3}\left(u_h^{m+2/3}-u_h^m,v_h\right)+\nu\left(\left(\theta u_h^m+(1-\theta)\,u_h^{m+2/3},v_h\right)\right)_h\\&+\hat{b}_h\left(u_h^{m+2/3},\theta u_h^m+(1-\theta)u_h^{m+2/3},v_h\right)=\left(f^{m+2/3},v_h\right),\qquad\forall\, v_h\in W_h.\end{aligned}\tag{13}$$

2. We compute the unique solution $u_h^{m+4/3} \in V_h$ of the linear problem

$$\begin{aligned}\frac{1}{4k/3}\left(u_h^{m+4/3} - u_h^m, v_h\right) + \nu\left(\left(\theta u_h^{m+4/3} + (1-\theta)u_h^m, v_h\right)\right)_h \\ = \left(f^{m+4/3}, v_h\right) - \hat{b}_h\left(u_h^m, u_h^m, v_h\right), \qquad \forall\, v_h \in V_h.\end{aligned} \tag{14}$$

3. We set

$$u_h^{m+1} = \frac{1}{2}\left(u_h^{m+2/3} + u_h^{m+4/3}\right). \tag{15}$$

This framework stands in particular for the usual finite difference and finite element techniques that are found in the literature (see [1]; see, also, [6] for a discussion on several important computational aspects).

3. THE MAIN RESULT

In the sequel, it will be assumed that the previous external approximations for $H_0^1(\Omega)^n$ and V satisfy four consistency hypotheses (here, $\mathcal{H}' \subset \mathcal{H}$, $\mathcal{H}' \to 0$, and $k' \to 0$).

HYPOTHESIS H1. *If $u_{h'} \in L^2(0,T;V_{h'})$, $\forall\, h' \in \mathcal{H}'$,*

$$u_{h'} \to u \text{ weakly in } L^2\left(0,T;L^2(\Omega)^n\right), \qquad \text{and}$$
$$p_{h'}u_{h'} \to \psi \text{ weakly in } L^2(0,T;F),$$

then necessarily $u \in L^2(0,T;V)$ and $\psi = \bar{\omega}u$.

HYPOTHESIS H2. *If $v_{h'}, w_{h'} \in L^2(0,T;W_{h'})$, $\forall\, h' \in \mathcal{H}'$,*

$$p_{h'}v_{h'} \to \bar{\omega}v \text{ weakly in } L^2(0,T;F), \qquad \text{and}$$
$$p_{h'}w_{h'} \to \bar{\omega}w \text{ strongly in } L^2(0,T;F),$$

then

$$\int_0^T ((v_{h'}(t), w_{h'}(t)))_{h'}\, dt \to \int_0^T ((v(t), w(t)))\, dt.$$

HYPOTHESIS H3. *If $u_{h'}, v_{h'} \in L^2(0,T;W_{h'})$, $\forall\, h' \in \mathcal{H}'$,*

$$p_{h'}u_{h'} \to \bar{\omega}u \text{ weakly in } L^2(0,T;F),$$
$$u_{h'} \to u \text{ strongly in } L^2\left(0,T;L^2(\Omega)^n\right),$$
$$p_{h'}v_{h'} \to \bar{\omega}v \text{ weakly in } L^2(0,T;F),$$

and $\psi_{k'} \to \psi$ strongly in $L^\infty(0,T)$, then necessarily

$$\int_0^T \hat{b}_{h'}\left(u_{h'}(t), v_{h'}(t), \psi_{k'}(t) r_{h'} w\right) dt \to \int_0^T \hat{b}\left(u(t), v(t), \psi(t) w\right) dt,$$

for all $w \in J(\Omega)$. We also assume that

$$\hat{b}_h(v_h, w_h, w_h) = 0, \qquad \forall\, v_h, w_h \in W_h, \quad \forall\, h \in \mathcal{H}, \tag{16}$$

and

$$\left|\hat{b}_h(u_h, v_h, w_h)\right| \le d_1 \|u_h\|_h \, \|v_h\|_h \, \|w_h\|_h,$$

where d_1 is independent from h.

HYPOTHESIS H4. *If $v_{h'} \in L^2(\mathbb{R}; V_{h'})$, $v_{h'} = 0$ a.e. in $\mathbb{R} \setminus [0,T]$, and for all $h' \in \mathcal{H}$, one has*

$$\int_0^T \|v_{h'}(t)\|_{h'}^2 \, dt \le \text{ const. } and$$
$$\int_0^T |\tau|^{2\beta} \, |\hat{v}_{h'}(\tau)|^2 \le \text{ const. } for\ some\ \beta > 0,$$

then $\{v_{h'}\}$ is precompact in $L^2(0,T; L^2(\Omega)^n)$. Here, $\hat{v}_{h'}$ is the Fourier transform of $v_{h'}$.

In the finite-dimensional space W_h, $\|\cdot\|_h$, and $|\cdot|$ are equivalent norms. More precisely, one has

$$\frac{1}{d_0}|w_h| \le \|w_h\|_h \le S(h)|w_h|, \qquad \forall\, w_h \in W_h, \tag{17}$$

for some "optimal" constants d_0 and $S(h)$. Obviously, $S(h)$ cannot be bounded as $h \to 0$ and its order of magnitude depends on the specific approximation spaces W_h. Thus, we can write

$$\left|\hat{b}_h(v_h, v_h, w_h)\right| \le S_1(h)|v_h|^2 \|w_h\|_h, \qquad \forall\, v_h, w_h \in W_h, \tag{18}$$

for some "optimal" $S_1(h)$ (evidently, $S_1(h) \le d_1 S(h)^2$). For given k and $h \in \mathcal{H}$, let us introduce the functions u_{kh}, v_{kh}, w_{kh}, $\tilde{u}_{kh}$, $\tilde{v}_{kh}$, and $\tilde{w}_{kh}$, which are defined as follows:

$$u_{kh}, v_{kh}, w_{kh} : [0,T] \to W_h \text{ are constant in each } [mk, (m+1)k)\,,$$
$$u_{kh}(t) = u_h^m, \qquad v_{kh}(t) = u_h^{m+2/3}, \qquad w_{kh}(t) = u_h^{m+4/3} \text{ in } [mk, (m+1)k)\,,$$
$$\tilde{u}_{kh}, \tilde{v}_{kh}, \tilde{w}_{kh} : [0,T] \to W_h \text{ are continuous and linear on each } [mk, (m+1)k)\,,$$
$$\tilde{u}_{kh}(mk) = u_h^m, \qquad \tilde{v}_{kh}(mk) = u_h^{m+2/3}, \qquad \tilde{w}_{kh}(mk) = u_h^{m+4/3}.$$

THEOREM 1. *Assume $\theta \in (0, 1/2)$. There exist constants c_0 and c_1 which depend only on*

$$|u_0|, \ \|f\|_{L^2(0,T;H)}, \ \nu, \ d_0, \ d_1, \text{ and } \theta$$

such that, if k and h satisfy

$$kS(h)^2 \le c_0, \qquad kS_1(h)^2 \le c_1, \tag{19}$$

then the following.

(a) *There exist $\mathcal{H}' \subset \mathcal{H}$, $\mathcal{H}' \to 0$, and $\{k'\} \to 0$ such that the corresponding subsequences $u_{k'h'}$, $v_{k'h'}$, $w_{k'h'}$, $\tilde{u}_{k'h'}$, $\tilde{v}_{k'h'}$, and $\tilde{w}_{k'h'}$ converge strongly in the space $L^2(0,T; L^2(\Omega)^n)$ and weakly-* in $L^\infty(0,T; L^2(\Omega)^n)$ towards the same function u. The associate sequences $p_h u_{k'h'}, \dots, p_{h'} \tilde{w}_{k'h'}$ converge weakly in $L^2(0,T; F)$ towards $\bar{\omega} u$.*

(b) *If $\mathcal{H}'$ and $\{k'\}$ satisfy all requirements in (a), then the common limit u is a solution to (6).*

(c) *Consequently, when $n = 2$, the whole sequences $u_{kh}, \dots, \tilde{w}_{kh}$ converge towards the unique solution.*

(d) *Finally, if $n = 2$ and*

$$kS(h)^2 \to 0, \qquad kS_1(h)^2 \to 0, \tag{20}$$

then the sequences $p_h u_{kh}, \dots, p_h \tilde{w}_{kh}$ are strongly convergent in $L^2(0,T; F)$.

Notice that (19) can be viewed as a stability condition. It is well known that these conditions appear frequently in the context of nonlinear parabolic problems. In practice, (19) means that, for small $|h|$, k cannot be too large—usually, $S(h)$ grows like $|h|^{-p}$ as $h \to 0$. Thus, Theorem 1 shows that (8)–(11) is (at least) conditionally stable. When (19) is satisfied, this algorithm leads to functions which are approximations to the solutions of the Navier-Stokes problem (6). For similar results concerning other algorithms, see [6,7].

4. PROOF OF THE MAIN RESULT

In order to demonstrate Theorem 1, we argue as in [1,6]. The proof will consist of several steps.

1st Step. "*A priori*" estimates for $u_{kh}, \dots, \tilde{w}_{kh}$.

2nd Step. Uniform estimates for the Fourier transform $\hat{u}_{kh}$ (and, consequently, for the norm of $\tilde{u}_{kh}$ in $H^\beta(0,T;L^2(\Omega)^n)$ for some β). For simplicity, the index h will be omitted in the first two steps.

3rd Step. The choice of a convergent subsequence.

4th Step. The proof that limit points solve (6).

5th Step. Strong convergence (when $n=2$ and (20) is satisfied).

The structure of this demonstration is common to other nonlinear problems and other fractional step methods. This becomes clear from the results in [1,6]. In a future work, we will present some general abstract results in this direction.

FIRST STEP. Let us take $v = \theta u^m + (1-\theta)u^{m+2/3} \in W$ in (13). Then,

$$\begin{aligned}\left|u^{m+2/3}\right|^2 - |u^m|^2 + (1-2\theta)\left|u^{m+2/3}-u^m\right|^2 + \frac{4k\nu}{3}\left\|\theta u^m + (1-\theta)\,u^{m+2/3}\right\|^2 \\ = \frac{4k}{3}\left(f^{m+2/3}, \theta u^m + (1-\theta)u^{m+2/3}\right).\end{aligned} \tag{21}$$

On the other hand, using $v = u^{m+4/3} \in V$ in (14), we find

$$\begin{aligned}&\left|u^{m+4/3}\right|^2 - |u^m|^2 + \left|u^{m+4/3}-u^m\right|^2 + \frac{8k\nu\theta}{3}\left\|u^{m+4/3}\right\|^2 \\ &+\frac{8k\nu(1-\theta)}{3}\left(\left(u^m, u^{m+4/3}-u^m\right)\right) + \frac{8k\nu(1-\theta)}{3}\|u^m\|^2 \\ &+\frac{8k}{3}\hat{b}\left(u^m,u^m,u^{m+4/3}\right) = \frac{8k}{3}\left(f^{m+4/3}, u^{m+4/3}\right).\end{aligned} \tag{22}$$

From (21) and (17), one has

$$\begin{aligned}&\left|u^{m+2/3}\right|^2 - |u^m|^2 + (1-2\theta)\left|u^{m+2/3}-u^m\right|^2 \\ &+\frac{2k\nu}{3}\left\|\theta u^m + (1-\theta)u^{m+2/3}\right\|^2 \le \frac{2kd_0^2}{3\nu}\left|f^{m+2/3}\right|^2,\end{aligned} \tag{23}$$

and from (22), (16), and (18), one also obtains the following:

$$\begin{aligned}&\left|u^{m+4/3}\right|^2 - |u^m|^2 + \frac{1}{2}\left|u^{m+4/3}-u^m\right|^2 + \frac{4k\nu\theta}{3}\left\|u^{m+4/3}\right\|^2 \\ &+\frac{8k\nu(1-\theta)}{3}\|u^m\|^2 - \frac{32}{9}k^2\left\{2S_1(h)^2|u^m|^2\|u^m\|^2\right. \\ &\left.+\,2\nu^2(1-\theta)^2S(h)^2\|u^m\|^2\right\} \le \frac{4kd_0^2}{3\nu\theta}\left|f^{m+4/3}\right|^2.\end{aligned} \tag{24}$$

On the other hand, we know from (15) that

$$2\left|u^{m+1}\right|^2 \le \left|u^{m+2/3}\right|^2 + \left|u^{m+4/3}\right|^2. \tag{25}$$

Thus, (23)–(25) together give

$$\begin{aligned}&\left|u^{m+1}\right|^2 - |u^m|^2 + \frac{1-2\theta}{2}\left|u^{m+2/3}-u^m\right|^2 + \frac{1}{4}\left|u^{m+4/3}-u^m\right|^2 \\ &+\frac{2k\nu\theta}{3}\left\|u^{m+4/3}\right\|^2 + \frac{4k\nu(1-\theta)}{3}\|u^m\|^2 \\ &+\frac{k\nu}{3}\left\|\theta u^m + (1-\theta)u^{m+2/3}\right\|^2 - \frac{32}{9}k^2\left\{S_1(h)^2|u^m|^2\|u^m\|^2\right. \\ &\left.+\,\nu^2(1-\theta)^2S(h)^2\|u^m\|^2\right\} \le \frac{kd_0^2}{3\nu}\left|f^{m+2/3}\right|^2 + \frac{2kd_0^2}{3\nu\theta}\left|f^{m+4/3}\right|^2.\end{aligned} \tag{26}$$

Recall that

$$\theta a^2 + (\theta a + (1-\theta)b)^2 \geq \frac{\theta}{2}a^2 + \frac{1-\theta}{4}b^2, \qquad \forall\, a, b \in \mathbb{R}.$$

Accordingly, (26) can also be written as follows:

$$\begin{aligned}
&\left|u^{m+1}\right|^2 - \left|u^m\right|^2 + \frac{1-2\theta}{2}\left|u^{m+2/3} - u^m\right|^2 + \frac{1}{4}\left|u^{m+4/3} - u^m\right|^2 \\
&+\frac{7k\nu\theta}{12}\left\|u^{m+4/3}\right\|^2 + \frac{k\nu(8-9\theta)}{6}\left\|u^m\right\|^2 + \frac{k\nu(1-2\theta)}{12}\left\|u^{m+2/3}\right\|^2 \\
&+k\left\{\frac{\nu\theta}{6}\left\|u^{m+1}\right\|^2 - \frac{32}{9}k\left[S_1(h)^2\left|u^m\right|^2\left\|u^m\right\|^2 + \nu^2(1-\theta)^2 S(h)^2\left\|u^m\right\|^2\right]\right\} \\
&\qquad\leq \frac{kd_0^2}{3\nu}\left|f^{m+2/3}\right|^2 + \frac{2kd_0^2}{3\nu\theta}\left|f^{m+4/3}\right|^2.
\end{aligned} \tag{27}$$

Now, if r is an integer with $0 \leq r \leq N$, summing up all the previous inequalities (27) for $m = 0, 1, \dots, r$, one is led to

$$\begin{aligned}
&\left|u^{r+1}\right|^2 + \frac{1-2\theta}{2}\sum_{m=0}^{r}\left|u^{m+2/3} - u^m\right|^2 + \frac{1}{4}\sum_{m=0}^{r}\left|u^{m+4/3} - u^m\right|^2 \\
&\quad+\frac{7k\nu\theta}{12}\sum_{m=0}^{r}\left\|u^{m+4/3}\right\|^2 + \frac{k\nu(8-9\theta)}{6}\sum_{m=0}^{r}\left\|u^m\right\|^2 \\
&\quad+\frac{k\nu(1-2\theta)}{12}\sum_{m=0}^{r}\left\|u^{m+2/3}\right\|^2 + k\left\{\frac{\nu\theta}{6}\sum_{m=0}^{r}\left\|u^{m+1}\right\|^2\right. \\
&\left.-\frac{32}{9}k\left[S_1(h)^2\sum_{m=1}^{r}\left|u^m\right|^2\left\|u^m\right\|^2 + \nu^2(1-\theta)^2 S(h)^2\sum_{m=1}^{r}\left\|u^m\right\|^2\right]\right\} \\
&\quad\leq \frac{kd_0^2}{3\nu}\sum_{m=0}^{r}\left|f^{m+2/3}\right|^2 + \frac{2kd_0^2}{3\nu\theta}\sum_{m=0}^{r}\left|f^{m+4/3}\right|^2 + \left|u^0\right|^2 \\
&\quad+\frac{32}{9}k^2 S_1(h)^2\left|u^0\right|^2\left\|u^0\right\|^2 + \frac{32}{9}k^2\nu^2(1-\theta)^2 S(h)^2\left\|u^0\right\|^2 \equiv \xi_r.
\end{aligned} \tag{28}$$

For convenience, let us set

$$\Phi(c_0, c_1, \theta) = \left|u_0\right|^2\left[1 + \frac{32}{9}c_0 c_1\left|u_0\right|^2 + \frac{32}{9}\nu^2(1-\theta)^2 c_0^2\right] + \frac{d_0^2}{2\nu}\|f\|^2_{L^2(0,T;H)}\left[1 + \frac{1}{\theta}\right].$$

Then, it is not difficult to check that, if (19) is satisfied with c_0 and c_1 being such that

$$(1-\theta)^2\nu^2 c_0 + c_1\Phi(c_0, c_1, \theta) \leq \frac{3\gamma\theta}{64}, \tag{29}$$

one also has

$$\xi_r \leq \xi_N \leq \Phi(c_0, c_1, \theta).$$

At present, we are going to prove that (28) is still true after dropping out the term between brackets. For this, we use induction in the following.

(a) For $r = 0$, it suffices to write (27) with $m = 0$.
(b) Assume our assertion is true for $m = 0, 1, \dots, r-1$. In particular,

$$\left|u^m\right|^2 \leq \xi_m \leq \Phi(c_0, c_1, \theta), \qquad \forall\, m \leq r,$$

and

$$\frac{\nu\theta}{6}\sum_{m=0}^{r}\|u^{m+1}\|^2-\frac{32}{9}k\left[S_1(h)^2\sum_{m=1}^{r}|u^m|^2\|u^m\|^2+\nu^2(1-\theta)^2S(h)^2\sum_{m=1}^{r}\|u^m\|^2\right]$$
$$\geq\sum_{m=1}^{r}\|u^m\|^2\left[\frac{\nu\theta}{6}-\frac{32}{9}\Phi(c_0,c_1,\theta)c_1-\frac{32}{9}\nu^2(1-\theta)^2c_0\right],$$

a positive quantity provided (19) is satisfied and (29) holds for c_0 and c_1. This proves that our assertion is also true for $m=r$.

Hence, one has

$$\begin{aligned}&\left|u^{r+1}\right|^2+\frac{1-2\theta}{2}\sum_{m=0}^{r}\left|u^{m+2/3}-u^m\right|^2+\frac{1}{4}\sum_{m=0}^{r}\left|u^{m+4/3}-u^m\right|^2\\&+\frac{7k\nu\theta}{12}\sum_{m=0}^{r}\left\|u^{m+4/3}\right\|^2+\frac{k\nu(8-9\theta)}{6}\sum_{m=0}^{r}\|u^m\|^2\\&+\frac{k\nu(1-2\theta)}{12}\sum_{m=0}^{r}\left\|u^{m+2/3}\right\|^2\leq\Phi(c_0,c_1,\theta),\end{aligned}$$

for all $r=0,1,\dots,N-1$. This leads to the following "*a priori*" estimates:

$$\begin{aligned}&\max_{0\leq m\leq N}|u^m|^2+\sum_{m=0}^{N}\left|u^{m+2/3}-u^m\right|^2+\sum_{m=0}^{N}\left|u^{m+4/3}-u^m\right|^2\\&+k\sum_{m=0}^{N}\|u^m\|^2+k\sum_{m=0}^{N}\left\|u^{m+2/3}\right\|^2+k\sum_{m=0}^{N}\left\|u^{m+4/3}\right\|^2\leq M,\end{aligned}\tag{30}$$

with M being a constant which depends only on $|u_0|$, $\|f\|_{L^2(0,T;H)}$, ν, c_0, and c_1. From (30), one deduces at once uniform estimates for u_k, v_k, w_k, $\tilde{u}_k$, $\tilde{v}_k$, and $\tilde{w}_k$:

$$\begin{aligned}&\|u_k\|_{L^\infty(0,T;L^2(\Omega)^n)}+\|u_k\|_{L^2(0,T;W_h)}+\|pu_k\|_{L^2(0,T;F)}\\&\leq\text{ const., the same holds for } v_k,\ w_k,\ \tilde{u}_k,\ \tilde{v}_k,\text{ and }\tilde{w}_k,\end{aligned}\tag{31}$$

$$\|u_k-v_k\|_{L^2(0,T;L^2(\Omega)^n)}\leq\text{const. }\sqrt{k},\text{ the same holds for } u_k-w_k,\ u_k-\tilde{u}_k,\text{ etc.}\tag{32}$$

SECOND STEP. Let us assume that $\tilde{u}_k$ is extended by zero and let us denote again by $\tilde{u}_k$ its extension. From (13)–(15), one obtains

$$\begin{aligned}&\frac{3}{k}\left(2u^{m+1}-2u^m,v\right)+\nu\left(\left(2(2-\theta)u^m+2(1-\theta)u^{m+2/3}+4\theta u^{m+4/3},v\right)\right)\\&+2\hat{b}\left(u^{m+2/3},\theta u^m+(1-\theta)u^{m+2/3},v\right)+4\hat{b}\left(u^m,u^m,v\right)\\&=\left(2f^{m+2/3}+4f^{m+4/3},v\right),\qquad\forall v\in V.\end{aligned}\tag{33}$$

This can also be written in the form

$$3\frac{d}{dt}\left(\tilde{u}_k(t),v\right)=\left(\left(g_k(t),v\right)\right),\qquad\forall v\in V,\quad t\text{ a.e. in }[0,T].\tag{34}$$

In (34), $g_k:[0,T]\to V$ is given by the equalities

$$\begin{aligned}((g_k(t),v))=&-\nu\left(\left((2-\theta)u_k(t)+(1-\theta)v_k(t)+2\theta w_k(t),v\right)\right)\\&-\hat{b}\left(v_k(t),\theta u_k(t)+(1-\theta)v_k(t),v\right)-2\hat{b}\left(u_k(t),u_k(t),v\right)\\&+\left(f_{1k}(t)+2f_{2k}(t),v\right),\qquad\forall v\in V,\end{aligned}$$

and each $f_{ik} : [0,T] \to H$ $(i = 1,2)$ is constant on $[mk,(m+1)k)$, with

$$f_{1k}(t) = f^{m+2/3}, \quad f_{2k}(t) = f^{m+4/3}, \qquad \forall t \in [mk,(m+1)k).$$

It is not difficult to prove that g_k is uniformly bounded in $L^1(0,T;V)$. Consequently, the arguments in [1] yield an estimate for the Fourier transform $\hat{u}_k$:

$$\int_{-\infty}^{\infty} |\tau|^{2\beta} \left|\hat{u}_k(\tau)\right|^2 d\tau \leq \text{ const. for some } \beta \in \left(0, \frac{1}{4}\right) \tag{35}$$

(for more details, see [6,7]).

THIRD STEP. Using (31) and the first consistency Hypothesis H1, one deduces that, for some $\mathcal{H}' \subset \mathcal{H}$ with $\mathcal{H}' \to 0$ and some $\{k'\} \to 0$, one has

$$\begin{aligned} \tilde{u}_{k'h'} \to u \text{ weakly-* in } L^\infty\left(0,T;L^2(\Omega)^n\right), \qquad &\text{and} \\ p_{h'}\tilde{v}_{k'h'} \to \bar{\omega} u \text{ weakly in } L^2(0,T;F),& \end{aligned} \tag{36}$$

as $h' \in \mathcal{H}'$, $h' \to 0$, $k' \to 0$, where $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$. On the other hand, (35) and H4 allow us to choose $\mathcal{H}'$ and k' such that, besides (36), the following is satisfied:

$$\tilde{u}_{k'h'} \to u \text{ strongly in } L^2\left(0,T;L^2(\Omega)^n\right). \tag{37}$$

Taking into account (32), the same convergence properties can be deduced for all other functions $v_{k'h'}, \dots, \tilde{w}_{k'h'}$.

FOURTH STEP. Choose $\varphi \in J(\Omega)$ and take $v = s_h\varphi$ in (33). If ψ is given with $\psi \in C^1([0,T])$ and $\psi(T) = 0$, it is readily found integrating by parts in $[0,T]$ that

$$\begin{aligned} -3\int_0^T (\tilde{u}_{kh}, \psi'(t)s_h\varphi)\,dt + \nu \int_0^T (((2-\theta)u_{kh}(t) + (1-\theta)v_{kh}(t) + 2\theta w_{kh}(t), \psi(t)s_h\varphi))_h\,dt \\ + \int_0^T \hat{b}_h\left(v_{kh}(t), \theta u_{kh}(t) + (1-\theta)v_{kh}(t), \psi(t)s_h\varphi\right)dt \\ +2\int_0^T \hat{b}_h\left(u_{kh}(t), u_{kh}(t), \psi(t)s_h\varphi\right)dt \\ = 3\left(\tilde{u}_{kh}(0), s_h\varphi\right)\psi(0) + \int_0^T (f_{1k}(t) + 2f_{2k}(t), \psi(t)s_h\varphi)\,dt, \end{aligned}$$

or in short form

$$-3I_{kh} + \nu J_{kh} + L_{kh} + 2M_{kh} = 3Q_{kh} + P_{kh}.$$

From Assumption (A4) (see Section 2) and (37), we see that

$$I_{k'h'} \to \int_0^T (u(t), \psi(t)\varphi)\,dt. \tag{38}$$

On the other hand, H2, H3, and (36),(37) written for $u_{k'h'}$ and $v_{k'h'}$ imply

$$\begin{aligned} \nu J_{k'h'} + L_{k'h'} + 2M_{k'h'} \to 3\nu \int_0^T ((u(t), \psi(t)\varphi))\,dt \\ +3\int_0^T \hat{b}\left(u(t), u(t), \psi(t)\varphi\right)dt. \end{aligned} \tag{39}$$

Also, $Q_{k'h'} \to (u_0, \varphi)\psi(0)$. This stems from (A4) and the fact that $u_h^0 \to u_0$ strongly in $L^2(\Omega)^n$ (recall that $\bigcup_h V_h$ is dense in $L^2(\Omega)^n$).

Notice that $f_{1k}+2f_{2k}\to 3f$ strongly in $L^2(0,T,L^2(\Omega)^n)$. Accordingly,

$$P_{k'h'}\to 3\int_0^T (f(t),\psi(t)\varphi)\;dt.$$

This, together with (38) and (39), leads to the following identity for u:

$$\begin{gathered}-\int_0^T (u(t),\psi'(t)\varphi)\;dt+\nu\int_0^T ((u(t),\psi(t)\varphi))\;dt\\ +\int_0^T \hat{b}\,(u(t),u(t),\psi(t)\varphi)\;dt=(u_0,\varphi)\psi(0)+\int_0^T (f(t),\psi(t)\varphi)\;dt.\end{gathered}\tag{40}$$

This must hold for any $\varphi\in J(\Omega)$, and any $\psi\in C^1([0,T])$ such that $\psi(T)=0$. By density, (40) has also to be true for all $\varphi\in V$. It is then straightforward to deduce that u solves (7). Finally, if $\psi\in\mathcal{D}([0,T])$ and $\psi(0)=1$, we find that

$$\begin{aligned}(u_0,\varphi)=-\int_0^T (u(t),\psi'(t)\varphi)\;dt+\nu\int_0^T ((u(t),\psi(t)\varphi))\;dt&\\ +\int_0^T \hat{b}\,(u(t),u(t),\psi(t)\varphi)\;dt-\int_0^T (f(t),\psi(t)\varphi)\;dt=(u(0),\varphi),&\end{aligned}$$

for all $\varphi\in V$ and the initial condition in (6) also holds. In other words, u solves the Navier-Stokes problem (6).

FIFTH STEP. Let us assume that $n=2$, $kS(h)^2\to 0$, and $kS_1(h)^2\to 0$, and let us prove that $p_k u_{kh}\to\bar{\omega}u$ strongly in $L^2(0,T;F)$ as $h\in\mathcal{H}$, $h\to 0, k\to 0$. With this purpose in mind, we introduce the functions $z_{kh}:[0,T]\to W_h$, which are supposed to be constant on each $[mk,(m+\ 1)k)$ and such that

$$z_{kh}(mk)=\left(\frac{1}{2}-\frac{\theta}{3}\right)u^{m+2/3}+\left(\frac{1}{2}+\frac{\theta}{3}\right)u^{m+4/3},\qquad\text{for all } m.$$

We also introduce, for each $h\in\mathcal{H}$, a function $u_h^+\in L^2(0,T;W_h)$ in such a way that $p_h u_h^+\to\bar{\omega}u$ strongly in $L^2(0,T;F)$ (see [1]). It will be convenient to use the following notation (here, $\alpha_{kh}\uparrow 1$ and β, δ are positive):

$$\begin{aligned}X_{kh}=2\left|u_h^N-u(T)\right|^2+\frac{1}{2}\sum_{m=0}^{N-1}\Bigg\{\frac{1}{2}\left|u^{m+4/3}-u^{m+2/3}\right|^2+(1-2\theta)\left|u^{m+2/3}-u^{m}\right|^2&\\ +\left|u^{m+4/3}-u^{m}\right|^2\Bigg\}+4\alpha_{kh}\nu\int_0^T\left\|z_{kh}(t)-u_h^+(t)\right\|^2\,dt&\\ -\beta\nu\int_0^T\|v_{kh}(t)-u_{kh}(t)\|^2\,dt-\delta\nu\int_0^T\|w_{kh}(t)-u_{kh}(t)\|^2\,dt.&\end{aligned}$$

Our argument is as follows. Assume that $u_{h'}^N$ converges weakly in $L^2(\Omega)^n$ towards a function χ as $h'\in\mathcal{H}'$, $h'\to 0$, and $k'\to 0$.

(i) It will be seen that

$$(\chi-u(T),v)=0,\qquad\forall v\in H.\tag{41}$$

(ii) Furthermore, it will be seen that, for small $|h|$ and k,

$$\begin{aligned}\frac{1}{2}\sum_{m=0}^{N-1}\left\{\frac{1}{2}\left|u^{m+4/3}-u^{m+2/3}\right|^2+(1-2\theta)\left|u^{m+2/3}-u^{m}\right|^2+\left|u^{m+4/3}-u^{m}\right|^2\right\}&\\ \geq\beta\nu\int_0^T\|v_{kh}(t)-u_{kh}(t)\|^2\,dt+\delta\nu\int_0^T\|w_{kh}(t)-u_{kh}(t)\|^2\,dt,&\end{aligned}\tag{42}$$

hence $X_{kh}\geq 0$.

(iii) Using (41) and (42), we shall prove that, for convenient choices of α_{kh}, β, and δ,

$$X_{k'h'} \to 0 \text{ as } h' \in \mathcal{H}', \qquad h' \to 0 \text{ and } k' \to 0, \tag{43}$$

in particular,

$$\int_0^T \left\| z_{k'h'} - u_{h'}^+ \right\|_{h'}^2 \, dt \to 0,$$

and $p_{h'}u_{k'h'} \to \bar{\omega}u$ strongly in $L^2(0,T;F)$. Since this argument can be applied to any subsequence for which $u_{h'}^N$ converges, we finally deduce that the whole sequence $\{p_h u_{kh}\}$ converges strongly.

Let us first check that (41) and (42) imply (43). For simplicity, index h will be omitted again. Notice that

$$X_k = X_k^1 + X_k^2 + X_k^3,$$

where

$$X_k^1 = 2|u(T)|^2 + 4\alpha_k \nu \int_0^T \left\|u^+(t)\right\|^2 \, dt,$$

$$X_k^2 = -4\left(u^N, u(T)\right) - 8\alpha_k \nu \int_0^T \left(\left(z_k(t), u^+(t)\right)\right) \, dt,$$

and

$$X_k^3 = X_k - X_k^1 - X_k^2.$$

It is immediate that

$$X_{k'}^1 \to 2|u(T)|^2 + 4\nu \int_0^T \|u(t)\|^2 \, dt$$

and

$$X_{k'}^2 \to -4|u(T)| - 8\nu \int_0^T \|u(t)\|^2 \, dt.$$

On the other hand, from (13) and (14) for $m = 0, 1, \dots, N-1$, using the identities

$$\left|u^{m+2/3}\right|^2 + \left|u^{m+4/3}\right|^2 = 4\left|u^{m+1}\right|^2 - 2\left(u^{m+2/3}, u^{m+4/3}\right),$$

we find

$$\begin{aligned} 3\left|u^N\right|^2 - 3\left|u^0\right|^2 + \frac{3}{2}\sum_{m=0}^{N-1}\Bigg\{ \frac{1}{2}\left|u^{m+4/3} - u^{m+2/3}\right|^2 \\ + (1-2\theta)\left|u^{m+2/3} - u^m\right|^2 + \left|u^{m+4/3} - u^m\right|^2 \Bigg\} \\ + k\nu \sum_{m=0}^{N-1} \Bigg\{ 2(1-\theta)\left\|u^{m+2/3}\right\|^2 + 2\left\|u^m\right\|^2 + 2(1+\theta)\left\|u^{m+4/3}\right\|^2 \\ - 2\theta(1-\theta)\left\|u^{m+2/3} - u^m\right\|^2 - 2(1-\theta)\left\|u^{m+4/3} - u^m\right\|^2 + \frac{4}{\nu}\hat{b}\left(u^m, u^m, u^{m+4/3}\right) \Bigg\} \\ = k\left\{ 2\left(f^{m+2/3}, (1-\theta)u^{m+2/3} + \theta u^m\right) + 4\left(f^{m+4/3}, u^{m+4/3}\right) \right\}. \end{aligned}$$

Accordingly, one sees that

$$\begin{aligned} X_k^3 = F_k + 2\left|u^0\right|^2 - \frac{1}{2}\sum_{m=0}^{N-1}\Bigg\{ \frac{1}{2}\left|u^{m+4/3} - u^{m+2/3}\right|^2 \\ + (1-2\theta)\left|u^{m+2/3} - u^m\right|^2 + \left|u^{m+4/3} - u^m\right|^2 \Bigg\} - \frac{4k\nu}{3}\sum_{m=0}^{N-1} p^m(\alpha_k, \beta, \delta, \theta). \end{aligned}$$

Here,

$$F_k = \frac{2k}{3}\sum_{m=0}^{N-1}\left\{2\left(f^{m+2/3},(1-\theta)u^{m+2/3}+\theta u^m\right)+4\left(f^{m+4/3},u^{m+4/3}\right)\right\}$$

and

$$\begin{aligned} p^m(\alpha_k,\beta,\delta,\theta) = {} & \|u^m\|^2 + \left(1+\theta(\alpha_k-1)-\frac{3}{2}\alpha_k\right)\left\|u^{m+2/3}\right\|^2 \\ & + \left(1+\theta(1-\alpha_k)-\frac{3}{2}\alpha_k\right)\left\|u^{m+4/3}\right\|^2 \\ & + \left(\frac{3}{4}\beta-\theta(1-\theta)\right)\left\|u^{m+2/3}-u^m\right\|^2 \\ & + \left(\frac{3}{4}\delta-1+\theta\right)\left\|u^{m+4/3}-u^m\right\|^2 \\ & + \alpha_k\left(\frac{3}{4}-\frac{\theta^2}{3}\right)\left\|u^{m+2/3}-u^{m+4/3}\right\|^2 + \frac{2}{\nu}\hat{b}\left(u^m,u^m,u^{m+4/3}\right). \end{aligned}$$

Notice that $F_k \to 4\int_0^T (f,u)\,dt$. It is also true that

$$\begin{aligned} X_k^3 = {} & F_k + A_k + 2\left|u^0\right|^2 \\ & - \frac{1}{2}\sum_{m=0}^{N-1}\left\{\frac{1}{2}\left|u^{m+4/3}-u^{m+2/3}\right|^2 + (1-2\theta)\left|u^{m+2/3}-u^m\right|^2 + \left|u^{m+4/3}-u^m\right|^2\right\} \\ & - \frac{4k\nu}{3}\sum_{m=0}^{N-1} q^m(\alpha_k,\beta,\delta,\theta), \end{aligned}$$

with

$$\begin{aligned} A_k &= \frac{4k\nu}{3}(1-a_k)\left[\left\|u^N\right\|^2 - \left\|u^0\right\|^2\right], \\ q^m &= p^m - (1-a_k)\left\|u^m\right\|^2 + (1-a_k)\left\|u^{m+1}\right\|^2, \end{aligned}$$

and

$$a_k = a_{kh} = \sqrt{k}S_1(h) \to 0.$$

Since by hypothesis $kS(h)^2 \to 0$, one also has $A_k \to 0$. If we prove that, by choosing α_k, β, and δ appropriately, $X_k^3 \le F_k + A_k + 2|u^0|^2$ for sufficiently small $|h|$ and k, then (43) will be demonstrated. Indeed, the previous inequality for X_k^3 yields

$$0 \le X_{k'} \le X_{k'}^1 + X_{k'}^2 + F_{k'} + A_{k'} + 2\left|u^0\right|^2,$$

and this right side converges to

$$2|u(0)|^2 - 2|u(T)|^2 - 4\nu\int_0^T \|u\|^2\,dt + 4\int_0^T (f,u)\,dt = 0$$

(it is essential here that $n = 2$, since we need the energy identity for u). Hence, it suffices to find α_k, β, and δ such that

$$\begin{aligned} -\frac{4k\nu}{3}q^m(\alpha_k,\beta,\delta,\theta) \le {} & \frac{1}{4}\left|u^{m+4/3}-u^{m+2/3}\right|^2 \\ & + \left(\frac{1-2\theta}{2}\right)\left|u^{m+2/3}-u^m\right|^2 + \frac{1}{2}\left|u^{m+4/3}-u^m\right|^2, \end{aligned} \tag{44}$$

for all m, when $|h|$ and k are small.

We know that

$$\begin{aligned}\frac{2}{\nu}\left|\hat{b}\left(u^m,u^m,u^{m+4/3}\right)\right| &= \frac{2}{\nu}\left|\hat{b}\left(u^m,u^m,u^{m+4/3}-u^m\right)\right| \\ &\le \frac{2}{\nu}S_1(h)\,|u^m|\,\|u^m\|\left|u^{m+4/3}-u^m\right| \\ &\le \frac{2d}{\nu}S_1(h)\,\|u^m\|\left|u^{m+4/3}-u^m\right| \\ &\le a_k\,\|u^m\|^2+\frac{d^2S_1(h)^2}{a_k\nu^2}\left|u^{m+4/3}-u^m\right|^2,\end{aligned} \tag{45}$$

with d being a constant. Let us take $\alpha_k = 1 - a_k$ (then $\alpha_k \uparrow 1$). From (45) and (15), one sees that

$$\begin{aligned}q^m \ge\, & a_k(1-\theta)\left\|u^{m+2/3}\right\|^2 + a_k(1+\theta)\left\|u^{m+4/3}\right\|^2 \\ &+(1-a_k)\left(\frac{1}{2}-\frac{\theta^2}{3}\right)\left\|u^{m+4/3}-u^{m+2/3}\right\|^2 \\ &+\left(\frac{3\beta}{4}-\theta(1-\theta)\right)\left\|u^{m+2/3}-u^m\right\|^2 \\ &+\left(\frac{3\delta}{4}-1+\theta\right)\left\|u^{m+4/3}-u^m\right\|^2-\frac{d^2S_1(h)^2}{a_k\nu^2}\left|u^{m+4/3}-u^m\right|^2.\end{aligned}$$

Thus, if β and δ are chosen such that $\beta > (4/3)\theta(1-\theta)$ and $\delta > (4/3)(1-\theta)$, then

$$-\frac{4k\nu}{3}\sum_{m=0}^{N-1} q^m(\alpha_k,\beta,\delta,\theta) \le \frac{4d^2}{3a_k\nu}kS_1(h)^2\sum_{m=0}^{N-1}\left|u^{m+4/3}-u^m\right|^2.$$

Since

$$\left(\frac{4d^2}{3a_k\nu}\right)kS_1(h)^2 \le \frac{1}{2},$$

for small $|h|$ and k, (44) must hold if β and δ are as before.

It remains only to prove (41) and (42). From (13)–(15), we easily find that

$$\begin{aligned}&\left(u_h^{m+1}-u_h^m,v_h\right)+\frac{k\nu}{3}\left(\left((2-\theta)u_h^m+(1-\theta)u_h^{m+2/3}+2\theta u_h^{m+4/3},v_h\right)\right)_h \\ &\quad+\frac{k}{3}\left[\hat{b}_h\left(u_h^{m+2/3},\theta u_h^m+(1-\theta)u_h^{m+2/3},v_h\right)+2\hat{b}_h\left(u_h^m,u_h^m,v_h\right)\right] \\ &\quad=\frac{k}{3}\left(f^{m+2/3}+2f^{m+4/3},v_h\right),\qquad \forall v_h\in V_h.\end{aligned}$$

Summing up for $m = 0, 1, \ldots, N-1$, one obtains the following equalities:

$$\begin{aligned}&\left(u_h^{N}-u_h^0,v_h\right)+\frac{k\nu}{3}\sum_{m=0}^{N-1}\left(\left((2-\theta)u_h^m+(1-\theta)u_h^{m+2/3}+2\theta u_h^{m+4/3},v_h\right)\right)_h \\ &\quad+\frac{k}{3}\sum_{m=0}^{N-1}\left[\hat{b}_h\left(u_h^{m+2/3},\theta u_h^m+(1-\theta)u_h^{m+2/3},v_h\right)+2\hat{b}_h\left(u_h^m,u_h^m,v_h\right)\right] \\ &\quad=\frac{k}{3}\sum_{m=0}^{N-1}\left(f^{m+2/3}+2f^{m+4/3},v_h\right),\qquad \forall v_h\in V_h.\end{aligned}$$

Now, using what is already known for the sequences u_{kh} and v_{kh}, and arguing as in the fourth step with $v_h = s_h v$ and $v \in J(\Omega)$, one can take limits and obtain

$$(\chi-u(0),v)+\nu\int_0^T((u(t),v))\,dt+\int_0^T b(u(t),u(t),v)\,dt=\int_0^T(f(t),v)\,dt.$$

This must hold for arbitrary $v \in J(\Omega)$. Obviously, this suffices to ensure that (41) is satisfied for all $v \in J(\Omega)$; since $J(\Omega)$ is dense in H, it must also be true for all $v \in H$. Finally, notice that (42) is implied by (17) and the fact that $kS(h)^2 \to 0$.

5. A NUMERICAL EXAMPLE

In this section, we describe briefly some numerical methods for the solution of the linear and nonlinear subproblems that are found after time discretization. To illustrate the behavior of algorithm (12)–(15), we also present some numerical results for the incompressible Navier-Stokes problem (6) in a particular 2-D domain (see Figure 1). In fact, we will consider here a situation in which the stationary state is quickly reached. Furthermore, the boundary conditions are not exactly as in Sections 1 and 2. So, other examples are probably more relevant, but the arguments apply as well.

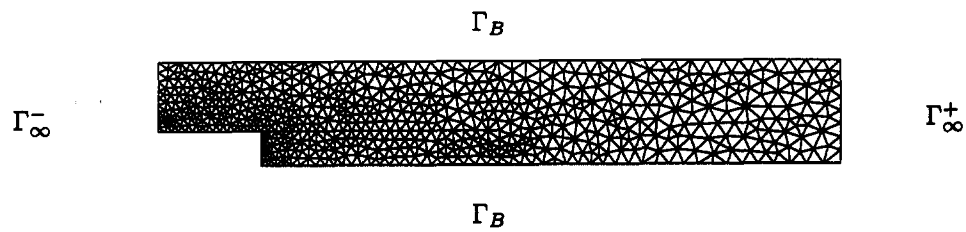

Figure 1.

The Numerical Solution of the Nonlinear Problems

For simplicity, we describe the numerical strategy in the framework of the "continuous" problem (9). Notice that this can be written as a Dirichlet problem for a quasi-linear elliptic system and, also, as follows.

$$\text{To find } u \in H_0^1(\Omega)^n \text{ such that } F(u) = 0. \tag{46}$$

Here, $F : H_0^1(\Omega)^n \to H^{-1}(\Omega)^n$ is an appropriate C^1 mapping. For the solution of (46), we have used an inexact Newton method, more precisely, GMRES iterates (see [8]). Recall that Newton iterates for (46) read:

(a) solve the linear system

$$F'\left(u^j\right) w^j = -F\left(u^j\right),$$

(b) set

$$u^{j+1} = u^j + w^j.$$

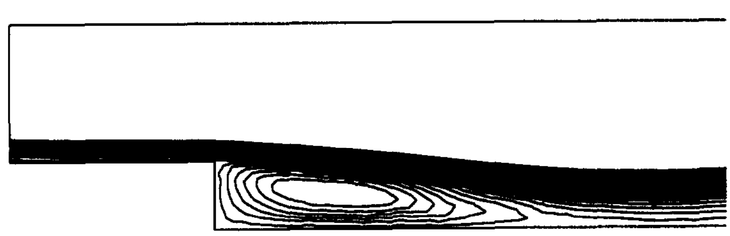

Figure 2.

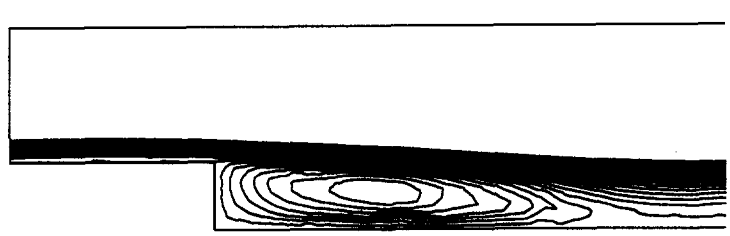

Figure 3.

Newton method is usually too expensive, mainly due to the fact that F' has to be computed at each step. This suggests the use of inexact Newton (or quasi-Newton) methods, where $F'(u^j)$ is replaced by an approximation. In particular, the GMRES (Generalized Minimum Residual) method of Saad and Schultz only needs, for each j, the knowledge of $F'(u^j)v$ for several v. This, in turn, can be approximated using the formula

$$F'\left(u^j\right)v \sim \frac{F\left(u^j+\sigma v\right)-F\left(u^j\right)}{\sigma},$$

where σ is chosen appropriately (for more details, see [8]; see, also, [9] for more information on the implementation of GMRES iterates in the context of the numerical simulation of fluid mechanics problems).

The Numerical Solution of the Linear Problems

There are, mainly, two families of methods for solving the quasi-Stokes (discretized) problems (12).

(a) Iterative methods (Uzawa and Arrow-Hurewicz algorithms; for instance, see [1]).

(b) Direct methods, for which one has to compute a complete set of basis functions in V_h (a particular direct method which is rather simple and at the same time accurate, reliant on the use of the nonconformal P_1 finite element of Crouzeix and Raviart; see [10]).

The Numerical Results

We compute a numerical approximation of the velocity field of a viscous incompressible fluid around a step (see Figure 1). The boundary $\partial\Omega$ of the fluid domain Ω is given by

$$\partial\Omega = \Gamma_B \cup \Gamma_\infty^- \cup \Gamma_\infty^+.$$

Here, Γ_B consists of the walls of the channel, and Γ_∞^- and Γ_∞^+ are the entering and exit boundaries, respectively. We have used the Crouzeix-Raviart finite element approximation which is determined by the triangulation in Figure 1 (this was generated by working with the MODULEF Library; see [11] for a description).

In our numerical experiments, we have taken Re $= 100$ and Re $= 191$ and a time discretization step $k = \Delta t = 0.1$. We have computed an approximation to the stationary solution, the asymptotic limit as $t \to +\infty$. The boundary conditions were the following: Poiseuille flow on Γ_∞^-, the no-slip condition on Γ_B and homogeneous natural conditions (free flow) on Γ_∞^+.

In Figure 2 and Figure 3, we present the streamlines of the computed solutions.

One observes that the length of the vortex is approximately $6L$ (when Re $= 100$) and $8L$ (when Re $= 191$), with L being the height of the step. This coincides with known experimental results (see [2]). For other numerical results and a more detailed study from a computational viewpoint (see [3]).

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